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Transport in random networks in a field: interacting particles

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Abstract. Transport through a random medium in a external field is modelled by particles performing biased random walks on the infinite cluster above the percolation threshold. Steps are more likely in the direction of the field—say downward—than against. A particle is allowed to move only onto an empty site (particles interact via hard core exclusion). Branches that predominantly point downwards and backbends—backbone segments on which particles must move upwards—act as traps. We have studied the movement of interacting random walkers in branches and backbends by Monte Carlo simulations and also analytically. In the full network, the trap-limited current flows primarily through the part of the backbone composed of paths with the smallest backbends and its magnitude in high fields is estimated. Unlike in the absence of interactions, the drift velocity does not vanish in finite fields. However, it continues to show a non-monotonic dependence on the field over a sizeable range of density and percolation probability.

1. Introduction

Particle transport in a disordered medium is often modelled by random walks on a random network [1]. The properties of walks are influenced strongly by the nature of connections in the random medium. For instance, if the disorder is such that the medium resembles a random fractal, the mean-squared displacement of an unbiased random walker grows anomalously slowly with time and the diffusion constant vanishes both in the absence [2] and presence [3] of interactions between walkers. However, if the disorder is milder and the large-distance connectivity properties remain similar to those of a homogeneous medium, then diffusion remains regular and qualitatively similar to that in the homogeneous case.

The application of an external field which induces a bias in a given direction leads to a new behaviour [4-20]. Even when disorder in the sense discussed above is mild, transport properties can be quite unusual and very different from those in the homogeneous medium. In this paper, we investigate the properties of an assembly of particles performing biased random walks on a random network, with hard core exclusion between particles.

Realisations of the problem may occur in various situations. One example is sedimentation in poorly connected porous materials. The directionality is imposed externally and globally: the field acts in a uniform manner on all parts of the network, e.g. gravity in the sedimentation problem. This is in contrast to systems in which the bias varies from one location to another, for example a current-dependent bias appropriate to hydrodynamic dispersion in porous media [21], or models with a random assignment of local directionality [22-25]. This latter case cannot in general be described in terms of a potential energy except in one dimension [24, 25] and produces effects different from those seen here.

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In related earlier work, Barma and Dhar [7] studied transport of non-interacting particles with a uniform external bias on the infinite cluster in percolation with bond occupation probability p. We continue to use this model of the disordered medium, which includes the homogeneous medium ($p = 1$), mild disorder ($p_c < p < 1$) and strong disorder ($p = p_c$) as special cases. They found that the drift velocity varies non-monotonically with the field for $1 > p > p_c$. The physical reason underlying this effect relates to the structure of the infinite percolation cluster (see figure 1): to paths which connect opposite faces of the medium (e.g. AB or CD) are attached branches of arbitrary length (e.g. PQ), which constitute dead-ends for particle transport. When a particle enters a branch which points predominantly along the field, it gets trapped for a long time. To exit from the branch and contribute to the flow, it must travel against the direction of the bias and this gets increasingly difficult as the field increases. Even on the backbone alone, particle motion is hindered in backbends, which are those portions of the backbone where particles must again travel against the direction of the bias (see HK in figure 1). Trapping effects are more pronounced in stronger fields and the competition between drift and trapping leads to the non-monotonicity discussed above. Furthermore, based on estimates for the occurrence of a trap (either a branch or a backbend) of length l (exponentially small in l), the net (macroscopic) distance moved, $R(t)$, was found to grow as t^k with $k < 1$ for bias exceeding a critical value [8, 9]. This implies an asymptotic, long-time drift velocity which vanishes in strong fields [7]. A vanishing mobility was also found by Ohtsuki [10] in a study of the motion of a single particle through the spaces between randomly packed hard spheres. Bottger and Bryskin [4] modelled hopping transport in disordered semiconductors by a percolation model and found that the conductivity remained non-zero for all finite fields. However, as discussed in § 6, their conclusions seem to be valid only when interparticle interactions are included.

In Monte Carlo studies of non-interacting biased random walkers on percolation clusters ($p > p_c$) the velocity v is found to vary non-monotonically with bias [12, 13]. The available data neither clearly confirm nor preclude a strong-field $v = 0$ regime,

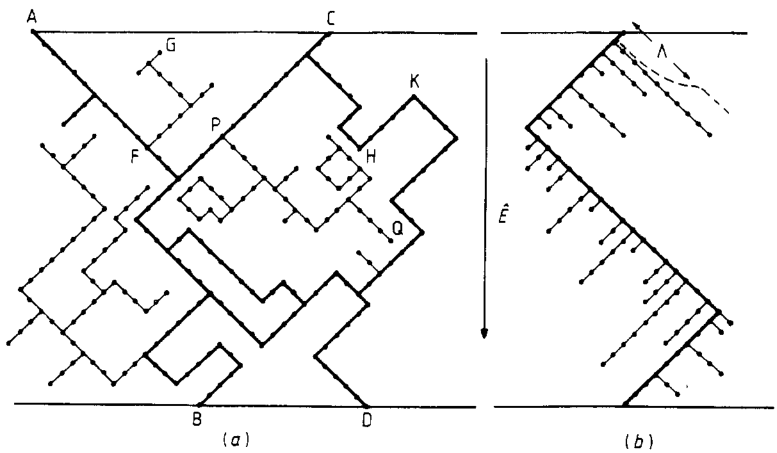

Figure 1. (*a*) A portion of the infinite percolation cluster is illustrated, with the backbone shown bold. The field acts downward and particles may get trapped in a branch like PQ or a backbend like HK. (*b*) The random comb is a model to illustrate branch-trapping. The broken curve is a sketch of the steady-state density profile in a branch. The length Λ marks the point beyond which the density is close to maximal.

though this does seem compatible. At $p = p_c$, Monte Carlo simulations showed $R(t)$ growing even slower than any power [14] consistent with the vanishing of k at criticality [8, 9]. In simulations with a topological bias (which acts outward from a given origin along network paths) the evidence for a transition to a $v = 0$ regime for $p > p_c$ is clearer [17]. With topological bias, branches act as traps while backbends do not. Thus topological bias would not be expected to mimic applied fields faithfully in situations in which backbends determine flow rates (as with interacting particles, see below).

In the present paper, we study the effects of interactions between particles: these are mutually repelling with hard core exclusion and move on the infinite percolation cluster in the presence of an external bias. Monte Carlo studies [19, 20] of this system indicate that the average displacement of particles behaves non-monotonically with the density of particles. In this paper, we focus on steady-state properties and find that, with interactions included, the drift velocity is always non-zero but shows non-monotonicities as functions of the field and density.

This paper is organised as follows. In the next section we describe the model system and the incorporation of interactions and bias. Facts about percolation that are germane to the present study are summarised. Section 3 deals with the trapping of particles in branches. In § 4, particle transport on backbends is described and the results of Monte Carlo simulations and an approximate treatment of the master equation are presented. Transport properties of the entire random network are examined in § 5, which is followed by a summary and discussion in § 6.

2. The model

2.1. Kinetics

In order to specify the configuration of the system, we must specify the occupation number n_i of each site i of the random network. Hard core exclusion forbids more than one particle on each site, so that n_i is either 0 or 1. The evolution of a configuration $\{n\}$ occurs through interchanges of particle-hole pairs (Kawasaki dynamics). The master equation fot the time dependence of the probability $\mathcal{P}(\{n\})$ of configuration $\{n\}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}(\{n\}) = \tfrac{1}{2}\sum_{i,j}[W_{ji}n_j\bar{n}_i\mathcal{P}(\{n\}_{ij}) - W_{ij}n_i\bar{n}_j\mathcal{P}(\{n\})]. \tag{1}$$

Configuration $\{n\}_{ij}$ is obtained from $\{n\}$ by replacing a particle (hole) in $\{n\}$ at site i by a hole (particle) and performing the same operation at a neighbouring site j, keeping site occupations on all other sites unaltered. $\bar{n}_i \equiv 1 - n_i$ is the number of holes on site i, and factors like $n_j\bar{n}_i$ arise in equation (1) as a particle interacts through hard core exclusion, and can hop only if there is a hole to accommodate it. W_{ij} is the probability per unit time that a particle at site i will be exchanged with a hole at site j, and is non-zero only if i and j are connected nearest-neighbour sites.

The external field is modelled by making the random walks biased, namely by making W_{ij} asymmetric. We assume that the field is oriented so that it has an equal component along every bond of the network. The hopping rate in the direction of the field (respectively against it) is $W(1+g)$ (respectively $W(1-g)$) where W is an elementary hopping rate unless the bond is missing (see below), in which case it is zero. The ratio of forward to backward hopping rates is related to the field strength E, inverse temperature β and lattice spacing a by

$$(1+g)/(1-g) = e^{\beta E a} \equiv e^{a/L(g)}. \tag{2}$$

This equation defines a bias-induced length $L(g)$ which plays an important role in the theory.

This system has been studied earlier [26] with attractive interactions between neighbouring particles (when interesting phase transitions occur in the current-carrying steady state), but on a non-random lattice. Also the velocity and diffusion constant have been computed exactly in one dimension [27].

2.2. *The random network*

As a model of the random network, we consider the infinite cluster in the percolation problem with bond occupation probability p. If the bond between nearest-neighbour sites i and j is absent, $W_{ij} = W_{ji} = 0$; if it is present, $W_{ij} = W(1 \pm g)$ and $W_{ji} = W(1 \mp g)$, the sign being chosen according to whether $\boldsymbol{E} \cdot (\boldsymbol{i}-\boldsymbol{j})$ is positive or negative.

As p varies from 1 (the pure lattice) to the critical percolation concentration p_c, the correlation length ξ varies from 0 to infinity. ξ is a measure of the linear extent of a typical finite cluster [28]. At $p=p_c$, the network is a random fractal. In the intermediate mild-disorder regime $p_c<p<1$, the network has the same connectivity properties as the pure system on length scales much greater than ξ.

One may distinguish between two types of sites on the infinite cluster. A site is on the backbone if there are at least two non-intersecting paths leading from it to infinity; otherwise it belongs to a branch—a cluster of sites attached to the backbone at a single point. Recognising that large branches are similar to large finite clusters, we expect that the probability of occurrence of a branch of linear extent $R \gg \xi$ is $\sim \exp(-R/\xi)$.

The external field imposes an overall directionality on the problem. It is further necessary to define backbends which are those segments of the backbone which correspond to backward excursions (with respect to the field) of a path connecting one side of the sample to the other. Backbends, called returns by Bottger and Bryskin [4], are important in the present problem as they limit the current that flows through the network in strong external fields ($g \lessdot 1$), as discussed later in §§ 4 and 5.

Above the directed percolation concentration p_d, there are backbend-free (directed) paths which span the sample [29]. Consider a (spanning) path with the constraint that every backbend is of length 1. One can define [30], analogous to p_d, a critical concentration $p_b(1)<p_d$, such that there are infinitely long paths satisfying this constraint. Similarly, one can find $p_b(l)$, which is the threshold for proliferation of paths with the constraint that backbends are of length at most l. It is clear that $p_b(l)$ has the limits $p_b(0)=p_d$ and $p_b(\infty)=p_c$. Alternatively, for given $p_c<p<p_d$, one can define a minimal backbend length $\zeta(p)$ as the smallest integer such that there are spanning paths with the constraint that every backbend on the path is at most of length $\zeta(p)$. For $p>p_d$, we have $\zeta(p)=0$, whereas $\zeta(p)$ diverges at $p \to p_c^+$. A heuristic argument [30] shows that $\zeta(p) \approx \text{constant} \times \xi$ as the percolation threshold p_c is approached.

Such backbend-constrained paths play a very important role in the transport of interacting particles. As will be discussed below, in very strong fields such paths are traversed in the shortest times (as opposed to typical 'chemically' shortest paths [31] on which arbitrarily long ($\gg \zeta(p)$) backbends occur, albeit exponentially rarely). On the Bethe lattice, $\zeta(p)$ has been calculated explicitly [30]. As $p \to p_c$, the ratio of lengths of the chemically shortest and ζ-constrained paths is expected to approach a constant ($\leqslant 1$). On the Bethe lattice, for instance, this ratio is $2/\pi$ independent of the coordination number.

3. Branch trapping

In studying the role played by branches in trapping particles, we consider only those branches which point predominantly in the direction of the field (e.g. branch PQ in figure 1, as opposed to FG). A simple model of random network on which some explicit calculations relating to branch trapping can be done is the random comb [9] (shown in figure 1) which consists of a linear backbone, from each site of which emanates a linear branch of random length. The field has equal components along the backbone and branches. Transport properties in the random comb may be expected to mimic those on the infinite cluster for $p > p_d$. But before turning to the comb as a whole, we study the motion of particles in a single branch.

3.1. Dynamics in a single branch

In the steady state, there is no current in the branch and the particles in the branch are described by the Hamiltonian

$$\mathcal{H} = -Ea \sum_{k=0}^{l} kn_k. \tag{3}$$

The site label k is 0 at the point where the branch is attached to the backbone and l at the bottom of the branch. On introducing the fugacity z, it is straightforward to evaluate the grand partition function

$$Z(z, g) = \prod_{k=0}^{l} (1 + z\,\mathrm{e}^{ka/L(g)}) \tag{4}$$

and the mean density $\rho_k \equiv \langle n_k \rangle$ at site k. On eliminating z in favour of the density ρ_0 at the attachment point, we have

$$\frac{\rho_k}{1-\rho_k} = \mathrm{e}^{ka/L(g)} \frac{\rho_0}{1-\rho_0}. \tag{5}$$

The density profile is sketched in figure 1. If $\rho_0 \ll 1$, the density ρ_k rises approximately exponentially until it reaches values of the order of unity. This happens at a depth approximately given by

$$\Lambda(\rho_0, g) = L(g) \ln\left(\frac{1-\rho_0}{\rho_0}\right). \tag{6}$$

For $ka \gg \Lambda$, the density is close to the saturation value of one particle per site.

Now consider fluctuations in which particles are driven out from the branch. Had there been no interactions between particles, a particle at the bottom of the branch would have had to surmount a potential barrier Ela to escape. The mean escape time in the non-interacting case is then given by Kramers' formula $\tau \sim \exp(la/L(g))$. This formula holds provided $la \gg L(g)$. In the presence of interactions, but with a fixed number n of particles in the branch, the first-passage time for all n particles to cross a site l' has been shown [32] to grow as $\tau \sim \exp n(l-l')/L(g)$ provided $n \ll L(g)/a \ll (l-l') \ll l$. However, the fixed number (of particles) constraint is not quite appropriate for a branch of the random comb; instead, the mean density ρ_0 at the attachment point should be specified. In this case, the typical time taken for a particle at site k to exit from the branch can be estimated from the ratio of the full grand partition function $Z(z, g)$ to a constrained grand partition function $Z'_k(z, g)$, where the constraint is that there are no particles above site k, except at site 0. (A similar procedure gave the correct asymptotic behaviour in the fixed-n case [32].) This yields

$$\tau_k \sim \frac{Z(z,g)}{Z'_k(z,g)} = \prod_{j=1}^{k} [1 + z\exp(ka/L(g))]. \tag{7}$$

The leading behaviour of τ_k is different depending on the relationship of the depth k to Λ:

$$\tau_k \sim \exp(ka/L(g)) \qquad \text{if } ka \ll \Lambda \tag{8a}$$

$$\tau_k \sim \exp(k^2 a/2L(g)) \qquad \text{if } ka \gg \Lambda. \tag{8b}$$

In the latter case, the density is so high that particles find it very hard to exit and remain in the branch for the most part.

3.2. *Transport through the random comb*

Motivated by the expected distribution of long branches in the infinite percolation cluster ($p > p_c$), we assume that a branch of length l on the random comb occurs with probability

$$\text{Prob}(l) = [1 - \exp(-a/\xi)^{-1}] \exp(-la/\xi). \tag{9}$$

We consider periodic boundary conditions for the backbone.

The mean time T_N taken by any one particle to go around the backbone is related to the current j_B in the backbone and the total number of particles $\mathcal{N}$ in the system by

$$T_N = \frac{\mathcal{N}}{j_B}. \tag{10}$$

This is because, in times longer than the emptying-out time of the longest branch in any particular finite realisation of the random comb, each particle will be recycled through every location of the comb several times and thus contributes equally to the current (see the discussion following equation (12) in [9]).

On an arbitrary network, an expression for the steady-state current between two neighbouring sites k and m is

$$j = W(1+g)\rho_k(1-\rho_m) - W(1-g)\rho_m(1-\rho_k) \tag{11}$$

where ρ_k, ρ_m are steady-state densities at sites k and m. (Two-site density correlations are neglected in this formula, which however is exact in one dimension [27].) On the random comb equation (11) leads to†

$$j_B = 2Wg\rho_0(1-\rho_0) \tag{12}$$

† Equation (12) is exact [33].

for the current on the backbone, where ρ_0 is the density of particles on the backbone. Furthermore, the total number of particles in the kth branch of length l_k is

$$n(l_k) = \sum_{k=0}^{l_k} \rho_k \tag{13}$$

where ρ_k is related to ρ_0 by equation (5). The total number of particles $\mathcal{N}$ in the system is then

$$\mathcal{N} = \sum_{k=1}^{N} n(l_k). \tag{14}$$

The drift velocity is defined as

$$v = \lim_{N\to\infty} \frac{Na}{T_N} \tag{15}$$

and is found, from equations (10)-(15), to be

$$v = \frac{2Wg\rho_0(1-\rho_0)}{\bar{n}}. \tag{16}$$

The limit $N \to \infty$ allows us to replace $\mathcal{N}/N$ by the configuration-averaged quantity

$$\bar{n} = \sum_l \mathrm{Prob}(l) n(l) \tag{17}$$

in equation (16).

The behaviour of v as a function of g is sketched in figures 2 and 3. The bold curve shows the result for non-interacting particles while the full curves show results for interacting particles. Figures 2 and 3 correspond to different boundary conditions.

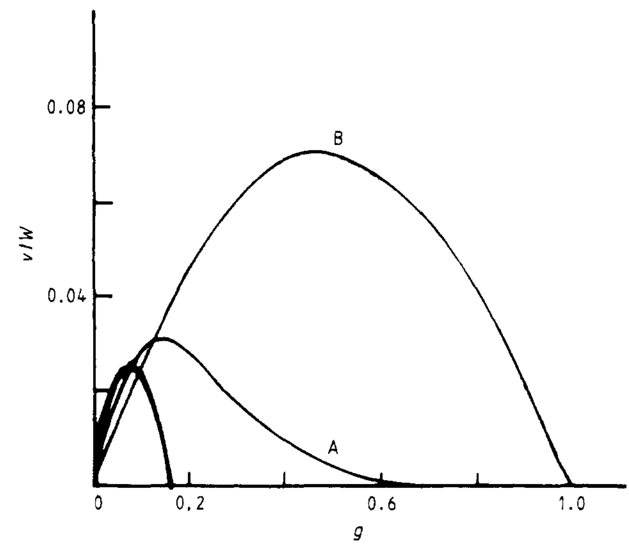

Figure 2. The drift velocity v in the random comb with $\xi = 3$ as a function of bias g, when the total number of particles is held fixed. The result for non-interacting particles (bold curve) is approached in the limit $\rho \to 0$. Curves A and B correspond to $\rho = 0.057$ and $\rho = 0.57$, respectively.

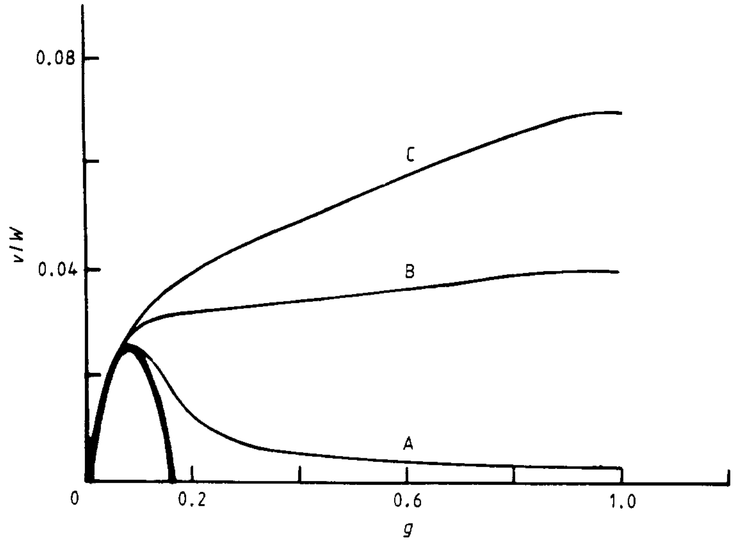

Figure 3. The drift velocity v in the random comb with $\xi = 3$ as a function of g when the backbone density ρ_0 is held fixed at 0.002 (A), 0.05 (B), 0.1 (C). The bold curve corresponds to the non-interacting particle limit $\rho_0 \to 0$.

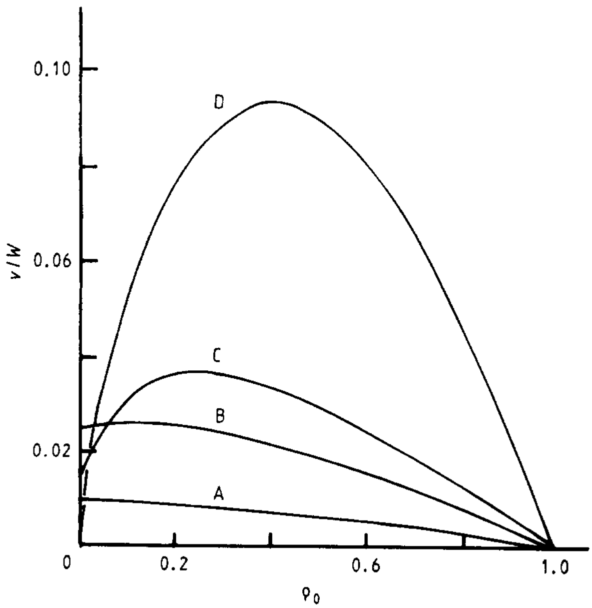

Figure 4. The drift velocity v in the random comb is plotted against the backbone density for several values of the bias. $g = 0.02$ (A), 0.08 (B), 0.14 (C), 0.50 (D).

Figure 2 shows results when the total number of particles $\mathcal{N}$ in the system is fixed (e.g. the system is closed, as with periodic boundary conditions on the backbone). Figure 3 shows results for a system which is open and in contact with a particle reservoir at either end in which case the density ρ_0 on the backbone remains fixed. In Figure 4 the variation of velocity as a function of the backbone density ρ_0 for a fixed value of the bias is shown. If the bias is large, v is a non-monotonic function of the filling, as was found in Monte Carlo simulations [19, 20].

4. Motion on the backbone

We focus here on the motion of particles on the backbone alone and disregard the effect of branches. (A discussion of transport on the full network is given in the next section.)

The backbone connects opposite faces of the sample but, for $p_c < p < p_d$, even the chemically shortest paths have backbends which act as traps and serve to limit the current. To simplify matters, we first study transport through a single backbend; this is cast as a problem involving interacting random walkers.

4.1. Dynamics in a single backbend

Consider a backbend of length l (e.g. HK in figure 1). There is a steady source of particles from the top to H and particles are continually drained off at K. This is modelled as biased diffusion of hard core particles in the segment $[0, l]$ of a 1D lattice, with the boundary conditions $\rho(0) = 1$ and $\rho(l) = 0$. The bias acts towards 0, but the boundary conditions force a current to flow from 0 to l.

The master equation that describes transport (equation (1)) is invariant under interchange of particles and holes and simultaneous relabelling of sites in reverse order, i.e. $n_k \to \bar{n}_{l-k}$. The boundary conditions respect this symmetry, implying that the steady-state density $\rho(k)$ at site k satisfies

$$\rho(k) = 1 - \rho(l - k). \tag{18}$$

Thus in the steady state the number of particles in the backbend is $l/2$ irrespective of the strength of the bias g. The principal effect of increasing g is to sharpen the region which marks the transition from the particle-rich half of the backbend to the hole-rich half. The steady-state profile approaches a step function centred at $k = l/2$ as $g \to 1$.

The current in the steady state is the number of particles crossing site l in unit time. We have investigated the dependence of the current j_l by Monte Carlo simulation. A single Monte Carlo step consists of considering l moves of the particles or holes, the sites being chosen at random. The bias acts in an opposite direction for a hole as compared to a particle. The results reported here are for averages over 2000 samples, and the dynamic evolution is followed for 7500–10 000 time units. Results for $(j_l/j_{l-n})^{1/n}$ are plotted in figure 5. For large l, the ratio is seen to approach $\exp(-a/2L(g))$, implying the asymptotic behaviour

$$j_l \sim \exp(-\tfrac{1}{2}la/L(g)). \tag{19}$$

In the steady state, an approximate expression for the current j is given by equation (11). The problem is to find the value of j for which the boundary conditions $\rho_0 = 1$,

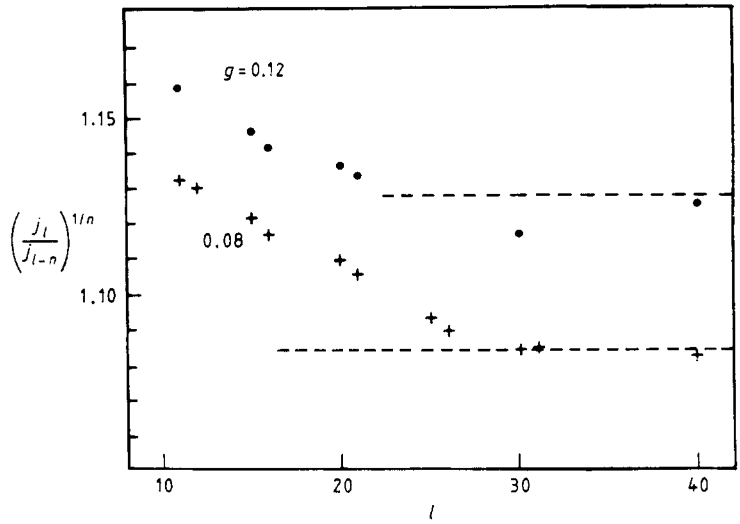

Figure 5. The nth root of the ratio of Monte Carlo determined currents through backbends of lengths l and $l-n$ plotted against l. We have taken $n=1$ for $l=10, 11$; $n=5$ for $l=14, 15$; $n=10$ for $l=20$ onwards. The ratio approaches $[(1+g)/(1-g)]^{1/2}$ as l increases.

$\rho_l=0$ hold. Defining y by

$$y_n = 1+g-2g\rho_n \tag{20}$$

we find the recurrence relation

$$y_{n+1} = 2 - A/y_n \tag{21}$$

with

$$A = (1-g^2) - 2gj/W. \tag{22}$$

In the continuum limit $(y_{n+1} \to y + \mathrm{d}y/\mathrm{d}x)$, equation (21) reduces to the differential equation

$$\mathrm{d}y/\mathrm{d}x = 2 - A/y. \tag{23}$$

Upon integration, and enforcing the boundary conditions $y_0 = 1-g$ and $y_l = 1+g$, this gives

$$(g^2+2jg)^{1/2} = g \coth \tfrac{1}{2}l(g^2+2jg)^{1/2}. \tag{24}$$

For long backbends $(l \gg 1)$, j is small and is given by

$$j \approx 2g\,\mathrm{e}^{-gl}. \tag{25}$$

This agrees with equation (19) when g is small $(L(g) \approx 1/2g)$, a necessary condition for the validity of the continuum approximation made above.

One can rationalise the result—in particular the factor $\frac{1}{2}$ in the exponent in equation (19)—as follows. The transport of a single particle through the backbend can be viewed as a parallel two-step process. In the first step, the topmost particle (located at site $k \approx l/2$ in large fields) is activated a distance $l/2$ from the top of the backbend. The energy cost is $\frac{1}{2}Ela$ and the associated activation time is $\tau_{l/2} \sim \exp(\frac{1}{2}la/L(g))$. In the second step, the consequent hole that remains in the steady-state distribution moves to the bottom and is filled up, by moving each of $l/2$ particles up a distance a. The energy cost is $\approx \frac{1}{2}Ela$ and the time required is $\tau_{1/2}$ again. In reality, of course, both processes occur in parallel, but the point is that they are (approximately) causally independent of each other and hence occur in a total time of order $\tau_{1/2}$ (not $\tau_{1/2}^2$). The current is then $\tau_{1/2}^{-1}$, and consequently follows equation (19).

4.2. *Transport on the backbone*

When $p_c < p < p_d$, any spanning path has several backbends of different lengths in series. On a single such path $\mathscr{P}$ of the backbone (ignoring parallel connections) the current $j(\mathscr{P})$ that could be supported by it is rate limited by the length $l^*(\mathscr{P})$ of the longest backbend (just as in a sequence of chemical reactions, the overall reaction rate depends on the slowest process). If $L(g) \ll l^*(\mathscr{P})$, we have

$$j(\mathscr{P}) \sim \exp(-cal^*(\mathscr{P})/L(g)) \tag{26}$$

where c is a constant of order unity.

If $\mathscr{P}$ is a typical (chemically) shortest path of the network, it has backbends of all lengths l with a probability $\sim \exp(-l/\tilde{\xi}(p))$, where $\tilde{\xi}$ is constant $\times\, \xi$. In a path of length aN, the likely largest l is $l^* \sim \tilde{\xi}$ ln N, while the time taken to cross the backbend of length l^* is given by the inverse of equation (26). Thus the time required to cover a macroscropic distance aN along a chemically shortest path grows as

$$T_N \sim N^x \tag{27}$$

with $x \propto \xi/L(g)$. If $\xi \gg L(g)$ (strong fields), the time increases faster than linearly, implying a vanishing current in the limit $N \to \infty$ along a single typical shortest path of the network.

However the current through the full backbone does not vanish. To see this, consider paths characterised by a largest allowed backbend length $\zeta(p)$ (see § 2.2). Since the length of every backbend on such a path is bounded by $\zeta(p)$, the current is finite and is given by equation (26) with $l^*(p) = \zeta(p)$. The union of all such 'ζ paths' forms a finite fraction of the backbone and would be expected to carry most of the current in strong fields; portions with backbends of length much greater than $\zeta(p)$ contribute comparatively little to the current and may be neglected in the strong-field limit.

For $p > p_d$, we have $\zeta(p) = 0$ and the primary current-carrying network is the directed backbone. As p approaches p_c, on the other hand, the minimal backbend length $\zeta(p)$ is proportional to the percolation correlation length ξ and diverges [30]. We then expect that as $p \to p_c$,

$$j \sim \exp\left(-\frac{b\beta E}{(p-p_c)^\nu}\right) \tag{28}$$

where the exponent ν characterises the divergence of ξ in that limit, and b is a constant of order unity.

5. Transport in the network

In a discussion of transport properties, it is necessary to distinguish between the current and the drift velocity as the two may behave differently. With suitable boundary conditions (spelt out below) the steady-state current J, i.e. the number of particles crossing unit area of the network per second, is determined primarily by the backbone. Branches carry no current, and their presence or absence does not affect J so long as the particle density on the backbone is maintained constant. This would be the case if, for instance, opposite faces of the sample were in contact with reservoirs with an inexhaustible supply of particles.

The drift velocity v, on the other hand, was defined in equation (10) as the length of the sample divided by the mean transit time. It measures the time spent by particles in the network, including excursions into side branches. Unlike the current, v is sensitive to the presence of branches.

We state this explicitly as there has been some controversy recently. In particular, Gefen and Goldhirsch [15] have defined a velocity $\tilde{v}$ for non-interacting particles on the random comb through the relation

$$J = \tilde{v}\bar{\rho}_{\mathrm{bb}} \tag{29}$$

where $\bar{\rho}_{\mathrm{bb}}$ is the mean particle density on the backbone. $\tilde{v}$ is a local velocity through backbone links and is the same whether or not there are branches, as both J and $\bar{\rho}_{\mathrm{bb}}$ have that property. In contrast to $\tilde{v}$, the drift velocity v is a macroscopic velocity and would be the appropriate quantity to compare with measurements on tagged particles moving over macroscopic distances in a porous medium. For instance, in the problem of hydrodynamic dispersion, v governs the mean location of dye particles [34]. The drift velocity satisfies a relation analogous to equation (29), namely

$$J = v\bar{\rho}. \tag{30}$$

This holds provided $\bar{\rho}$ is the mean density over *all* sites, including branch sites.

Returning to particles with hard core exclusion, we discuss the regimes $p_{\mathrm{c}} < p < p_{\mathrm{d}}$ and $p_{\mathrm{d}} < p < 1$ separately below.

5.1. $p_c < p < p_d$

All spanning connections have backbends on them and do not allow transmission of particles if $g = 1$ (infinitely strong field). Thus the current J and the drift velocity v both vanish in this limit.

In large but finite fields, the current in the backbone drops (using equations (25) and (26)) as a power of $(1-g)$:

$$J \propto (1-g)^{b\zeta(p)}. \tag{31}$$

The exponent, which is proportional to the minimum backbend length required for infinitely long connections, diverges as $p \to p_{\mathrm{c}}$. Furthermore, the way in which v vanishes as $g \to 1$ is similar to equation (31).

In the intermediate and low g regimes, the behaviour is expected to be similar to that in the non-interacting case, namely a linear rise at low g with slope proportional to diffusion constant, followed by a drop (see figure 6(a)).

5.2. $p_d < p < 1$

Directed connections now span the sample and the single most important difference with the previous case is that now the current is non-zero as $g \to 1$. In this limit, the mean time spent in a branch approaches a finite value, while it takes infinitely long to traverse a backbend. Consequently the current is carried only by the directed backbone, i.e. the backbone of the infinite cluster in the directed percolation problem.

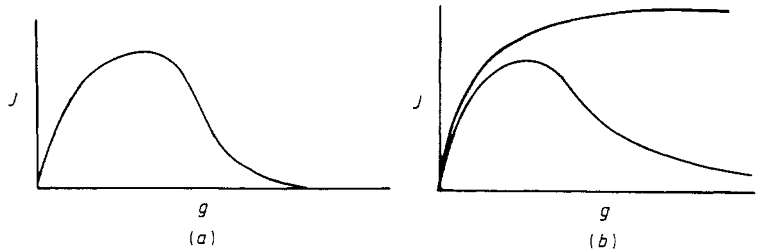

Figure 6. The current against bias (schematic) when (*a*) p is between p_c and p_d, and (*b*) when p is larger than p_d. In the latter case, the $g = 1$ current does not vanish, but the dependence on field is monotonic only if p is large enough.

The mobility of each particle through directed paths is quite high; at low density, the single-particle result $2W$ is approached. However, for p just larger than p_d, the current J is low as connections are sparse. The low-flux current J is proportional to the number of inlets per unit area on the face (or any other cross section) of the sample. Thus we expect

$$J(g=1) \propto P_{BB}^{dir} \tag{32}$$

where P_{BB}^{dir} is the fraction of sites in the directed backbone. As $J(g=1)$ is small for $p \gtrsim p_d$, the dependence of J of g continues to be non-monotonic (figure 6(*b*)). As p increases, so does $J(g=1)$, and eventually the non-monotonicity as a function of g is lost (figure 6(*b*)). The drift velocity is expected to be qualitatively similar to that in the random comb and, in particular, to exhibit the non-monotonic behaviour apparent in figure 2 when the density of particles is fixed.

6. Conclusion

For non-interacting particles, both branches and backbends act as traps, as the field tends to push particles towards local minima of potential energy. The primary effect of interparticle interactions is to reduce the accessibility, and hence the effectiveness, of traps. A particle at the bottom of a long branch takes much longer to escape when there are hard core interactions than when there are none (equation (8)). But the branch bottom is inaccessible to most particles on the backbone because of intervening particles in the branch. The combination of extreme inaccessibility and extremely long trapping times (in the rare event that a particle is trapped) leads to a finite mean time spent in a branch. Thus the drift velocity is finite, as illustrated by the exact calculation on the random comb in § 3.

The notion of limited accessibility applies to backbends too. Consequently, the current is rate-limited by ζ, the minimal backbend length required to form infinite connections.

The question arises: how low must the density be in order that results resemble those in the absence of interactions? The answer is that the interaction-induced length $\Lambda(\rho_0, g) \equiv L(g) \ln(1/\rho_0)$ introduced in § 3 must be much larger than the correlation length ξ. This condition guarantees that in most branches the density is low enough so that essentially single-particle dynamics operates. Moreover, close to p_c typical backbend lengths are also proportional to ξ; the condition $\Lambda \gg \xi$ ensures that the time between arrivals of successive particles exceeds that required by a single particle to clear a typical backbend, implying particles do not accumulate in such backbends. Of course, even when $\Lambda \gg \xi$, very long branches and backbends (with depths $\gg \Lambda$) do fill up with particles and become much less accessible and effective as traps. Consequently, the velocity remains finite (though small) in the region $L(g) < \xi$ whereas it would have vanished for non-interacting particles.

In very strong fields ($\Lambda \ll \xi$) almost all backbends and branches are full and thus relatively inaccessible. A large fraction of the current is then carried by the sub-backbone composed of paths on which every backbend is smaller than some minimal length $\zeta(p)$. This length vanishes above p_d and most of the current is carried by the directed backbone.

In their discussion of disordered semiconductors, Bottger and Bryskin envisage transport through paths in which all backbend lengths are bounded. However, it would seem that their conclusions are valid for interacting particles and not for non-interacting Boltzmann particles, which is the case they address (see equation (4) of [4]). For non-interacting particles which sample traps of all depths the high-field mobility on the percolation cluster vanishes [7-9] and the ζ backbone plays no special role. If the current is to be channelled through the ζ backbone, interactions are essential in order to limit accessibility to backbends and branches with yet longer trapping times.

We have addressed only steady-state properties in this paper. However, the approach to the steady state is likely to be slow—perhaps anomalously so—as backbends both on the backbone and in branches must be surmounted before the steady-state density is established in the full network.

It would be interesting to have an experimental test of these predictions, in particular of the non-monotonic dependence of the velocity on the bias. The assumed short-ranged (essentially contact) nature of the interactions may place some restrictions on the types of systems that could be described by this treatment. The effects discussed here might be observed in experiments (like sedimentation) involving particle motion through randomly porous media.

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